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Fractional Order GN Model on Thermoelastic Interaction in an Infinite Fibre-Reinforced Anisotropic Plate Containing a Circular Hole

Ibrahim A. Abbas

Faculty of Science and Arts-Khulais, Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia; Faculty of Science, Department of Mathematics, Sohag University, Sohag, Egypt

The present work is aimed at the study of thermoelastic interactions in an infinite fibre-reinforced anisotropic plate containing a circular hole in the context of a fractional Green and Naghdi theory of type II (fractional-type II) in which the model of heat conduction with time-fractional order. The inner surface of cavity is assumed to be stress free and is subjected to a thermal shock. The problem is solved numerically using the finite element method. According to numerical results and graphs, it is found that the introducing a fractional derivative of order α has a significant effect on the displacement, temperature and stresses. Some comparisons are made to estimate the effects of the presence and absence reinforcement.

Keywords: Fractional-Type II, Fiber-Reinforced, Finite Element Method.

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1. INTRODUCTION

During recent years, several interesting models have been developed by using fractional calculus to study the physical processes particular in the area of heat conduction, diffusion, viscoelasticity, mechanics of solids, control theory, electricity etc. It has been realized that the use of fractional order derivatives and integrals leads to the formulation of certain physical problems which is more economical and useful than the classical approach. There exists many material and physical situations like amorphous media, colloids, glassy and porous materials, manmade and biological materials/polymers, transient loading etc., where the classical thermoelasticity based on Fourier type heat conduction breaks down. In such cases, one needs to use a generalized thermoelasticity theory based on an anomalous heat conduction model involving time fractional (non integer order) derivatives. Green and Naghdi[1] in 1993 postulated a new concept in generalized thermoelasticity which is called the thermoelasticity without energy dissipation. The principal feature of this theory is that in contrast to the classical thermoelasticity associated with Fourier's law of heat conduction, the heat flow does not involve energy dissipation. In addition, the theory permits the transmission of heat as thermal waves at finite speeds. Green and Naghdi[2–4] as an alternate way of formulating the propagation of heat.

The generalization of the concept of derivative and integral to a non-integer order has been subjected to several approaches, and various alternative definitions of fractional derivatives appeared.[5–10] Among the few works devoted to applications of fractional calculus to thermoelasticity, we can refer to the works of Povstenko,[11–14] who introduced a fractional heat conduction law, found the associated thermal stresses. Sherief et al.,[15] Youssef[16] and Ezzat[17,18] introduced new models of thermoelasticity using a fractional heat conduction equation. Abbas[19] introduced another new model of a fractional heat conduction equation, which was developed the Green and Naghdi theory with time-fractional order.

Fiber-reinforced composites are used in a variety of structures due to their low weight and high strength. Materials such as resins reinforced by strong aligned fibers exhibit highly anisotropic elastic behavior in the sense that their elastic moduli for extension in the fiber direction are frequently of the order of 50 or more times greater than their elastic moduli in transverse extension or in shear. The theory of strongly anisotropic materials has been extensively discussed in the literature, Belfield et al.[20] studied the stress in elastic plates reinforced by fibers lying in concentric circles. Sengupta and Nath[21] discussed the problem of surface waves in fiber-reinforced anisotropic elastic media. Abbas and Abd-Alla[22] showed that, the effect of relaxation time in generalized thermoelastic interaction in an infinite fibre-reinforced anisotropic plate containing a circular hole. Tian et al.,[23] Abbas and Abbas et al.[24–27] applied the finite element method in different generalized

thermoelastic problems. Recently,[28–30] studied other problems in waves.

This paper considers thermoelastic with time-fractional derivative problem involving such circumferentially reinforced plates. The composite material is then locally transversely isotropic, with the direction of the axis of transverse isotropy now not constant, but everywhere directed along the tangents to circles in which the fibers lie. The problem has been solved numerically using a finite element method (FEM). The displacement, the temperature and stresses distributions are shown graphically with some comparisons.

2. PROBLEM FORMULATION: GOVERNING EQUATIONS

In the context of the Green and Naghdi theory,[1] abbas[19] and Abbas and Abd-Alla,[22] the field equations for linear equations governing fractional order thermoelastic interactions in a fiber-reinforced linearly thermoelastic anisotropic medium whose preferred direction is that of a unit vector $\boldsymbol{a}$, in the absence of body forces and heat sources, are as follows:

$$\sigma_{ij,j}=\rho u_i \quad i,j=1,2,3 \tag{1}$$

$$K^*T_{,ii}=\frac{\partial^{1+\alpha}}{\partial t^{1+\alpha}}(\rho c_e T+T_o\beta_{ij}u_{i,i}), \quad 0<\alpha\le 1, \quad i,j=1,2,3 \tag{2}$$

$$\begin{aligned}\sigma_{ij}=&\lambda e_{kk}\delta_{ij}+2\mu_T e_{ij}+\gamma(a_k a_m e_{km}\delta_{ij}+a_i a_j e_{kk})\\&+2(\mu_L-\mu_T)(a_i a_k e_{kj}+a_j a_k e_{ki})+\beta a_k a_m e_{km} a_i a_j\\&-\beta_{ij}(T-T_0)\delta_{ij}, \quad i,j,k,m=1,2,3\end{aligned} \tag{3}$$

where ρ is the mass density; u_i the displacement vector components; e_{ij} the strain tensor; σ_{ij} the stress tensor; T the temperature change of a material particle; T_o the reference uniform temperature of the body; β_{ij} the thermal elastic coupling tensor; c_e the specific heat at constant strain; K^* the material characteristic of the theory; λ, μ_T are elastic parameters; $\gamma, \beta, (\mu_L-\mu_T)$ are reinforced anisotropic elastic parameters and the component of the vector **a** are (a_1, a_2, a_3) where $a_1^2+a_2^2+a_3^2=1$. The comma notation is used for spatial derivatives and superimposed dot represents time differentiation. For circumferential reinforcement, it is normal to employ a system of cylindrical polar coordinates (r,θ,z) and henceforth all components are referred to these coordinates. In this system, for cylindrical symmetric interactions, the displacement vector possesses only the radial component $u=u(r,t)$, where r is the radial distance measured from the origin (point of symmetry), and the stress tensor is determined by the radial stress σ_{rr} and the circumferential stress (hoop stress) $\sigma_{\theta\theta}$. For circumferential reinforcement the vector $\boldsymbol{a}$ is everywhere directed in the tangential (i.e., θ) direction, so that in cylindrical polar coordinates $\boldsymbol{a}$ has components $(0,1,0)$. In this case, Eqs. (1)–(3) yield the following governing equations for u and T:

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{1}{r}(\sigma_{rr}-\sigma_{\theta\theta})=\rho\frac{\partial^2 u}{\partial t^2} \tag{4}$$

$$K^*\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}\right)=\frac{\partial^{1+\alpha}}{\partial t^{1+\alpha}}\left(\rho c_e T+T_o\beta_{11}\frac{\partial u}{\partial r}+T_o\beta_{22}\frac{u}{r}\right) \tag{5}$$

$$\sigma_{rr}=(\lambda+2\mu_T)\frac{\partial u}{\partial r}+(\lambda+\gamma)\frac{u}{r}-\beta_{11}(T-T_o) \tag{6}$$

$$\sigma_{\theta\theta}=(\lambda+\gamma)\frac{\partial u}{\partial r}+(\lambda+2\gamma+4\mu_L-2\mu_T+\beta)\frac{u}{r}-\beta_{22}(T-T_o) \tag{7}$$

with $\beta_{11}=2(\lambda+\mu_T)\alpha_{11}+(\lambda+\gamma)\alpha_{22}$, $\beta_{22}=2(\lambda+\gamma)\alpha_{11}+(\lambda+2\alpha+4\mu_L-2\mu_T+\beta)\alpha_{22}$, where α_{11}, α_{22} are coefficients of linear thermal expansion. For convenience, the following non-dimensional variables are used:

$$\begin{aligned}&(r',u')=\frac{1}{b}(r,u), \quad t'=\frac{c_1}{b}t,\\&(\sigma'_{rr},\sigma'_{\theta\theta})=\frac{1}{D}(\sigma_{rr},\sigma_{\theta\theta}), \quad c_1=\sqrt{\frac{D}{\rho}},\\&D=\lambda+2\gamma+4\mu_L-2\mu_T+\beta, \quad T'=\frac{T-T_o}{T_o}\end{aligned} \tag{8}$$

In terms of the non-dimensional quantities defined in Eq. (8), the above governing equations reduce to (dropping the dashed for convenience)

$$\frac{\partial}{\partial r}\left(B_1\frac{\partial u}{\partial r}+B_2\frac{u}{r}-B_3T\right)+(B_1-B_2)\frac{1}{r}\frac{\partial u}{\partial r}+(B_2-1)\frac{u}{r^2}-(B_3-B_4)\frac{T}{r}=\frac{\partial^2 u}{\partial t^2} \tag{9}$$

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}=\frac{\partial^{1+\alpha}}{\partial t^{1+\alpha}}\left(\varepsilon_1 T+\varepsilon_2\frac{\partial u}{\partial r}+\varepsilon_3\frac{u}{r}\right) \tag{10}$$

$$\sigma_{rr}=B_1\frac{\partial u}{\partial r}+B_2\frac{u}{r}-B_3T \tag{11}$$

$$\sigma_{\theta\theta}=B_2\frac{\partial u}{\partial r}+\frac{u}{r}-B_4T \tag{12}$$

where $(B_1,B_2,B_3,B_4)=\frac{1}{D}(\lambda+2\mu_T,\lambda+\gamma,T_o\beta_{11},T_o\beta_{22})$,

$$(\varepsilon_1,\varepsilon_2,\varepsilon_3)=\left(\frac{\rho c_e c_1^2}{k^*},\frac{c_1^2\beta_{11}}{k^*},\frac{c_1^2\beta_{22}}{k^*}\right)$$

The surface of the hole i.e., $r=b$ is assumed to be stress free and is subjected to a uniform step in temperature effect so that the boundary conditions are taken as:

$$\sigma_{rr}(b,t)=0, \quad T(b,t)=T_1H(t) \tag{13}$$

where $H(t)$ denotes the Heaviside unit step function.

Initially the medium is at rest and undisturbed and the initial conditions are:

$$u(r,0)=\dot{u}(r,0)=0, \quad T(r,0)=\dot{T}(r,0)=0 \tag{14}$$

2.1. Finite Element Method

The Finite element method is a powerful technique originally developed for numerical solution of complex problems in structural mechanics, and it remains the method of choice for complex systems. A further benefit of this method is that it allows physical effects to be visualized and quantified regardless of experimental limitations. In this section, the governing equations of generalized thermoelasticity with fractional order derivative based upon Green and Naghdi of type II are summarized, using the corresponding finite element equations. The finite element equations of a generalized thermoelasticity problem can be readily obtained by following standard procedure. In the finite element method, the displacement component u and temperature T are related to the corresponding nodal values by

$$u=\sum_{i=1}^{m} N_i u_i(t), \quad T=\sum_{i=1}^{m} N_i T_i(t) \tag{15}$$

where m denotes the number of nodes per element, and N the shape functions. In the framework of standard Galerkin procedure, the weighting functions and the shape functions coincide. Thus,

$$\delta u=\sum_{i=1}^{m} N_i \delta u_i, \quad \delta T=\sum_{i=1}^{m} N_i \delta T_i \tag{16}$$

with Eqs. (13) and (14), $u'=u_{,i}$ and $T'=T_{,i}$ can be expressed as

$$u'=\sum_{i=1}^{m} N_i' u_i(t), \quad T'=\sum_{i=1}^{m} N_i' T_i(t) \tag{17}$$

$$\delta u'=\sum_{i=1}^{m} N_i' \delta u_i, \quad \delta T'=\sum_{i=1}^{m} N_i' \delta T_i \tag{18}$$

Thus, the finite element equations corresponding to Eqs. (9)–(12) can be obtained as

$$\sum_{e=1}^{me}\left(\begin{bmatrix} M_{11}^e & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} \ddot{u}^e \\ \ddot{T}^e \end{Bmatrix}+\begin{bmatrix} 0 & 0 \\ C_{21}^e & C_{22}^e \end{bmatrix}\begin{Bmatrix} \dot{u}^e \\ \dot{T}^e \end{Bmatrix} + \begin{bmatrix} K_{11}^e & K_{12}^e \\ 0 & K_{22}^e \end{bmatrix}\begin{Bmatrix} u^e \\ T^e \end{Bmatrix}=\begin{Bmatrix} F_1^e \\ F_2^e \end{Bmatrix}\right) \tag{19}$$

where me is the total number of elements. The coefficients in Eq. (19) are given below.

$$M_{11}^e=\int [N]^T[N]dr$$

$$C_{21}^e=\int \frac{\partial^\alpha}{\partial t^\alpha}[N]^T\left(\varepsilon_2[N']+\frac{\varepsilon_3}{r}[N]\right)dr$$

$$C_{22}^e=\int \frac{\partial^\alpha}{\partial t^\alpha}\varepsilon_1[N]^T[N]dr$$

$$K_{11}^e=\int\left[[N']^T\left(B_1[N']+\frac{B_2}{r}[N]\right) +[N]^T\left(\frac{B_2-B_1}{r}[N']+\frac{1-B_2}{r^2}[N]\right)\right]dr$$

$$K_{12}^e=\int\left[-B_3[N']^T[N]+\frac{B_3-B_4}{r}[N]^T[N]\right]dr$$

$$K_{22}^e=\int\left[[N']^T[N]-\frac{1}{r}[N]^T[N]\right]dr,$$

$$F_1^e=[N]^T\bar{\tau}|_1^r, \quad F_2^e=[N]^T\bar{q}|_1^r$$

Symbolically, the discretized equations of Eq. (19) can be written as

$$M\ddot{d}+C\dot{d}+Kd=F^{\text{ext}} \tag{20}$$

where M, C, K and F^{ext} represent the mass, damping, stiffness matrices and external force vectors, respectively; $d=[u\ T]^t$; $\bar{\tau}$ represent the component of the traction, and $\bar{q}$ represents heat flux. On the other hand, the time derivatives of the unknown variables have to be determined by Newmark time integration method or other methods (see Wriggers[31]).

2.2. Numerical Example

To study the effect of reinforcement on wave propagation, we use the following physical constants for generalized fibre-reinforced thermoelastic materials.[22]

$$\rho=2660\ \text{kg/m}^3, \quad \lambda=5.65\times10^{10}\ \text{N/m}^2$$

$$\mu_T=2.46\times10^{10}\ \text{N/m}^2, \quad \mu_L=5.66\times10^{10}\ \text{N/m}^2$$

$$\alpha=-1.28\times10^{10}\ \text{N/m}^2, \quad \beta=220.90\times10^{10}\ \text{N/m}^2$$

$$\alpha_{11}=0.017\times10^{-4}\ \text{deg}^{-1}, \quad \alpha_{22}=0.015\times10^{-4}\ \text{deg}^{-1}$$

$$c_e=0.787\times10^3\ \text{J kg}^{-1}\ \text{deg}^{-1}, \quad T_o=293\ \text{k}$$

Before going to the analysis, the grid independence test has been conducted. The grid size has been refined and consequently the values of different Further refinement of mesh size over 10000 elements does not change the values considerably, which is therefore accepted as the grid size for computing purposes. Here all the variables/parameters are taken in non-dimensional forms. The results for displacement, temperature, radial stress and hoop stress has been carried out by taking $T_1=1$ and $t=0.3$.

The first group (Figs. 1–4) show the four curves predicted by the different theories of Green and Naghdi theory of Type II ($\alpha=1$) and fractional Green and Naghdi theory of Type II ($\alpha=0.2, 0.5, 0.8$) with reinforcement (WRE). As expect, the fractional order has a great effect on the distribution of field quantities.

The second group (Figs. 5–8) represent the variations of the physical quantities under fractional Green and Naghdi

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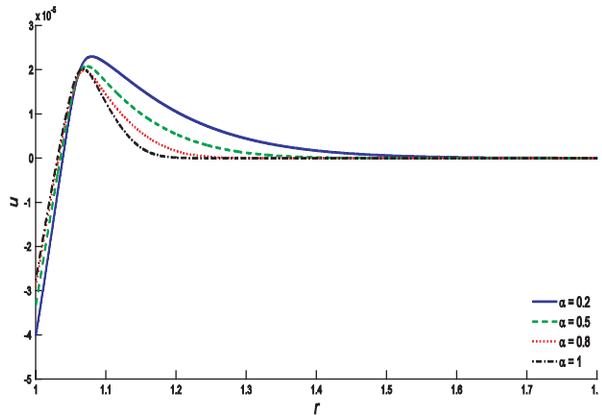

Fig. 1. Variation of u against r for different values of α at $t=0.3$.

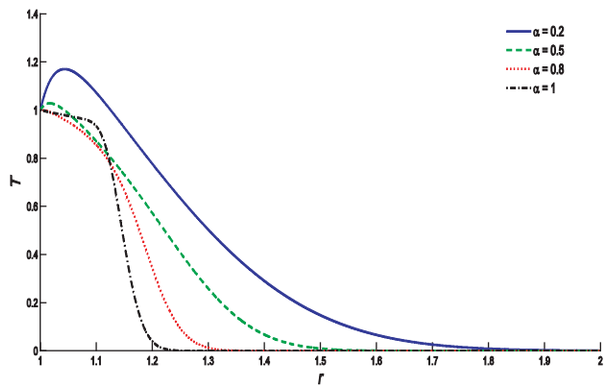

Fig. 2. Variation of T against r for different values of α at $t=0.3$.

theory of Type II ($\alpha=0.3$) with reinforcement (WRE) and without reinforcement (NRE). In Figures 5–8, the solid line (—) refers to a thermoelastic solid without reinforcement (NRE) and the dotted line (…) refers to a thermoelastic solid with reinforcement (WRE). Figure 5 represents the radial variations of displacement with reinforcement (WRE) and without reinforcement (NRE). It is observed that the displacement is continuous and the displacement gradually decreases with r and is zero at $r=1.7$ for (NRE), is zero at $r=1.4$ for (WRE). This is also in agreement

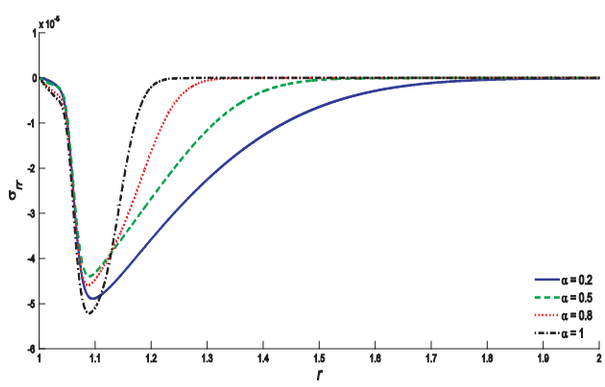

Fig. 3. Variation of σ_{rr} against r for different values of α at $t=0.3$.

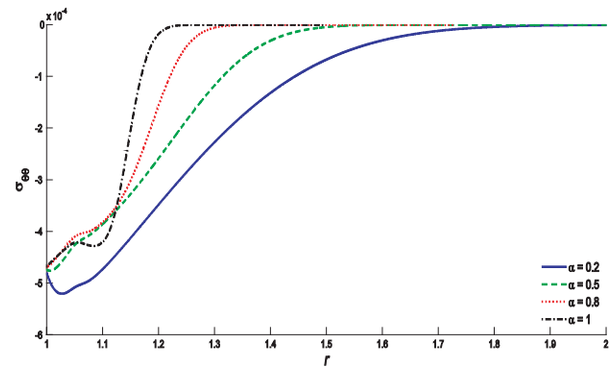

Fig. 4. Variation of $\sigma_{\theta\theta}$ against r for different values of α at $t=0.3$.

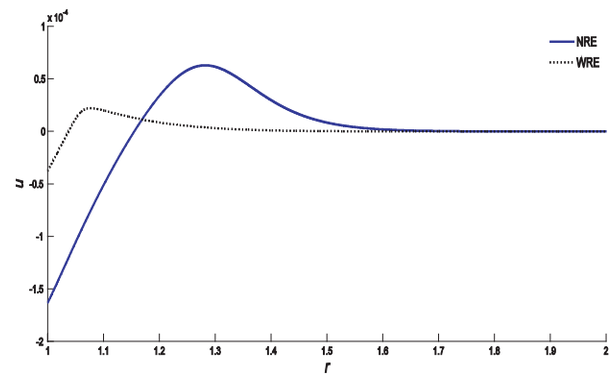

Fig. 5. Variation of u against r at $t=0.3$ and $\alpha=0.3$ with (WRE) and without (NRE) reinforcement.

with the theoretical result where beyond the thermal wave front, the displacement vanishes. From the result there is no significant difference in the value of the temperature that is observed with reinforcement (WRE) and without reinforcement (NRE) as shown in Figure 6. Figure 7 shows the graphical presentation of the radial stress versus distance r and indicates finite jumps at the elastic wave fronts and then it approaches and ultimately becomes zero. Figure 8 gives the variation of the hoop stress versus r.

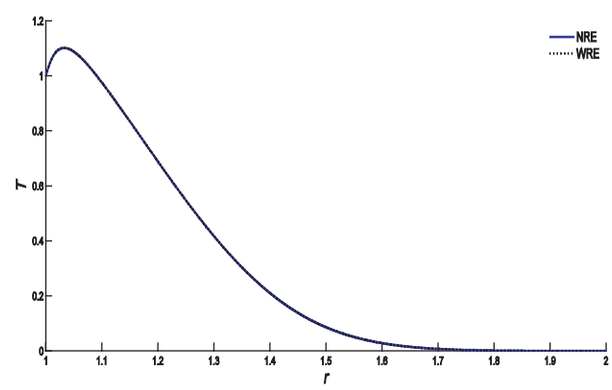

Fig. 6. Variation of T against r at $t=0.3$ and $\alpha=0.3$ with (WRE) and without (NRE) reinforcement.

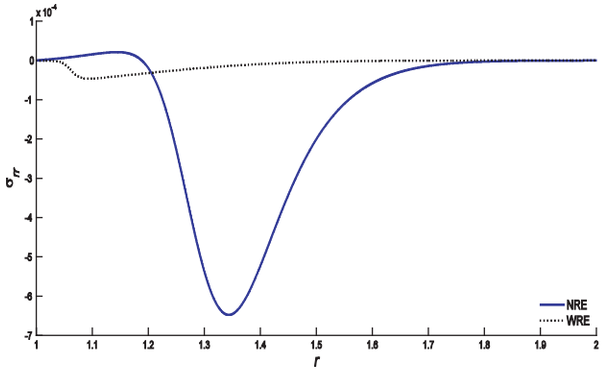

Fig. 7. Variation of σ_{rr} against r at $t=0.3$ and $\alpha=0.3$ with (WRE) and without (NRE) reinforcement.

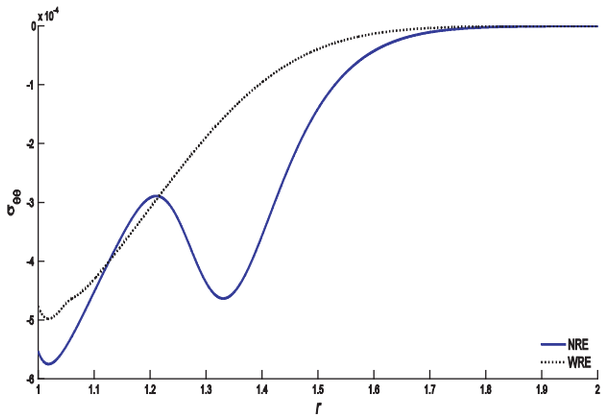

Fig. 8. Variation of $\sigma_{\theta\theta}$ against r at $t=0.3$ and $\alpha=0.3$ with (WRE) and without (NRE) reinforcement

The hoop stress at first decreases, and then suffers a finite jump at the elastic wave front, and then it approaches and ultimately becomes zero. We observed that the radial stress is zero at $r=1$ which satisfies the boundary conditions of the problem. Hence, we conclude with the following points.

(1) In all figures, it is noticed that the fractional order α has a significant effect on all the fields, and the curves are smoother in the case $(\alpha=0.2, 0.5, 0.8)$.

(2) The way from the result there is no significant difference in the value of temperature is noticed with reinforcement (WRE) and without reinforcement (NRE).

(3) The reinforcement has a great effect on the distribution of displacement and stresses.

3. CONCLUSION

In this paper, we have studied the fractional order Green and Naghdi model on thermoelastic interaction in an infinite fibre-reinforced anisotropic plate containing a circular hole. According to the results, we have to construct a new classification to all the materials according to its fractional parameter, where this parameter becomes new indicator of its ability to conduct the thermal energy.

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